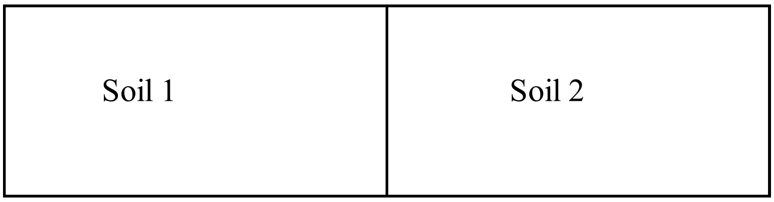
Semaine 5

Exercice 5_1

Assume steady horizontal flow of water in the one-dimensional unsaturated system (ψ is suction):



$x = 0$ $x = L/2$ $x = L$

$\Psi(0) = \psi_0$ $\Psi(L) = \psi_L$

For soil 1, $K_1(\psi) = K_{1s} \exp(-\alpha_1 \psi)$

For soil 2, $K_2(\psi) = K_{2s} \exp(-\alpha_2 \psi)$

(a) Starting from Darcy's Law (i.e., assuming steady flow), show that for 1D flow in a horizontal, thin, layer:

$$q_i = K_i(\psi)\frac{d\psi}{dx}, i = 1, 2$$

(b) Using the expression in (a), obtain an implicit equation from which the suction $\psi(L/2)$ can be calculated.

(c) If $\alpha_1 = 5$ m^{-1}, $K_{1S} = 10$ m y^{-1}, $\psi_0 = 4$ m, $\alpha_2 = 7$ m^{-1}, $K_{2S} = 14$ m y^{-1} and $\psi_L = 1$ m, show that, to one decimal place, $\psi(L/2) = 1.4$ m.

(d) In what direction is the flow taking place in (c)?

(e) What is the flux through the soil if L = 1 m?

Exercice 5_2

(a) Transform Richards' equation for horizontal, unsaturated flow:

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left[D(\theta)\frac{\partial\theta}{\partial x}\right], x > 0, t > 0$$

$$\theta(x, 0) = \theta_i, x > 0$$

$$\theta(0, t) = \theta_0, t > 0$$

into

$$-\frac{\phi}{2}\frac{d\theta}{d\phi} = \frac{d}{d\phi}\left[D(\theta)\frac{d\theta}{d\phi}\right]$$

$$\theta(0) = \theta_0$$

$$\theta(\infty) = \theta_i$$

using the Boltzmann transformation ($\phi = x/t^{1/2}$), i.e., $\theta(x,t) \rightarrow \theta(\phi)$.

(b) Show that, using the transformed equation in (a), that the capillary diffusivity, $D(\theta)$, is given by:

$$D(\theta) = -\frac{1}{2}\frac{d\phi}{d\theta}\int_{\theta_i}^{\theta}\phi(\bar{\theta})d\,\bar{\theta}$$

This is known as the **Bruce and Klute equation**

(c) From the result in (a), what can you deduce about horizontal unsaturated flow? Hint: Consider a laboratory experiment in which moisture profiles are measured at different times, i.e., from the experiment, a series of data sets in the form $\theta(x,t_1)$, $\theta(x,t_2)$, $\theta(x,t_3)$, etc., are known.

Exercice 5_3

Deux tensiomètres sont placés dans un sol sableux à une profondeur de 80 et 120 cm et indiquent des charges de pression valant respectivement -20 et -50 cm. Lors d'une seconde mesure effectuée quelques jours plus tard, les valeurs lues ont passé à -60 et 0 cm. Indiquer la direction des flux et évaluer leur intensité (dans les deux cas) à 1m de profondeur, sachant que la fonction de conductivité hydraulique de ce sol est décrite par la relation suivante :

$$K(h) = \frac{10^4}{10^3 + \left[(-10) - h\right]^3} \quad [\text{cm/j}]$$

dans laquelle la charge h est exprimée en cm. On supposera que la charge de pression varie linéairement entre les profondeurs d'implantation des tensiomètres.

Exercice 5_4 (extra challenge)

In terms of the matric tension, h ($= p/\rho g$), Richards' equation is written in the form (z is positive downwards):

$$\frac{d\theta}{dh}\frac{\partial h}{\partial t} = \frac{\partial}{\partial z}\left[K(h)\left(\frac{\partial h}{\partial z} - 1\right)\right]$$

where the symbols have their usual meanings. We wish to solve this equation for an initially dry soil with ponding at the soil surface. Thus take h at the surface to be $h(0,t) = p(0,t)/\rho g = h_S > 0$. Suppose that the water capacity term is given by:

$$\frac{d\theta}{dh} = \frac{-\alpha^*}{(h - h_s)}\frac{dK(h)}{dh}.$$

Now let $h = h_S + z/A(t)$, where $A(t)$ is a function to be determined. At $z = 0$, we always have $h = h_S$, in agreement with the surface boundary condition. In the following, you will use $h(z,t) \rightarrow h(z/A(t))$.

(a) Show that with this water capacity and similarity variable, Richards' equation reduces to:

$$\alpha^* A\frac{dA}{dt} = 1 - A$$

Hint: You need to transform the derivatives with respect to t and z remembering (z,t) is replaced by z/A(t).

(b) For A(0) = 0, show that the solution to the equation in (a) is

$$\frac{-t}{\alpha^*} = A(t) + \ln[1 - A(t)].$$

(c) Use Darcy's Law to derive an expression for the flux at any position and time, q(z,t). In particular, for a ponded surface condition show that:

$$q(0,t) = K_s\left[1 - \frac{1}{A(t)}\right],$$

where K_S is the saturated conductivity.

(d) The cumulative infiltration at the surface, I(t), is defined by:

$$I(t) = \int_0^t q(0,\bar{t})\, d\bar{t}.$$

Show that $I(t) = -\alpha^* K_S A(t)$.

(e) Show that the expression in (b) has the form of the Green-Ampt infiltration equation (The Green-Ampt model is a simplified, but very useful, infiltration model):

$$K_s t = I(t) - \frac{S^2}{2K_s}\ln\left[1 + \frac{I(t)}{S^2/(2K_s)}\right]$$

where

$$S^2 = 2\alpha^* K_s^2$$

You have not seen this in class, but for completeness, note that S is the soil's sorptivity, a property that quantifies the short-time infiltration of water into soil.